

CALCULATION OF THE COMPLEXITIES OF SUBSTITUTIVE SEQUENCES OVER A BINARY ALPHABET

BO TAN, ZHI-XIONG WEN, AND YIPING ZHANG

ABSTRACT. We consider the complexities of substitutive sequences over a binary alphabet. By studying various types of special words, we show that, knowing some initial values, its complexity can be completely formulated via a recurrence formula determined by the characteristic polynomial.

Keywords: Substitution, Special Word, Complexity

2000 Mathematics Subject Classification: Primary 20M05; Secondary 68R15.

1. INTRODUCTION

The study of substitutions over a finite alphabet plays important roles in many fields such as finite automata, symbolic dynamics, formal languages, number theory, fractal geometry *etc.* It has various applications to quasi-crystals, computational complexity, information theory... (see [1, 2, 7, 9, 10] and the references therein). In addition, substitutions are also fundamental objects in combinatorial group theory [11, 12].

Given an infinite sequence $\xi = \xi_1\xi_2\xi_3\cdots$ ($\xi_i \in \mathbb{A}$) over some finite alphabet \mathbb{A} , we denote by $\mathcal{L}_n(\xi)$ the set $\{\xi_i\cdots\xi_{i+n-1} \mid i \geq 1\}$ of factors of ξ of length n ($n \geq 1$), and by convention $\mathcal{L}_0(\xi)$ is the singleton consisting of the empty word ε . The set $\mathcal{L}(\xi) = \cup_{n \geq 0} \mathcal{L}_n(\xi)$ is then called the language of ξ , and the function $p_\xi(n) := \#\mathcal{L}_n(\xi)$ the complexity of ξ , here and hereafter $\#$ denotes the cardinality of a finite set.

Let \mathbb{A}^* be the free monoid generated by \mathbb{A} (with ε as the neutral element). A morphism $\sigma : \mathbb{A}^* \rightarrow \mathbb{A}^*$ is called a substitution. We deal with only the non-erasing substitutions (the image of any letter in \mathbb{A} is not the empty word), whence the substitution can be extended naturally to $\mathbb{A}^\mathbb{N}$, the set of infinite sequences over \mathbb{A} . Denote by ξ_σ any one of the fixed points of σ (that is $\sigma(\xi_\sigma) = \xi_\sigma$), if it exists.

The study of the complexity of ξ_σ (also called the complexity of σ) has a long history. In general, it is very difficult to find out the explicit formula for $p_\xi(n)$ for a given σ ; only some calculations for specific classes of substitutions can be found in the literature. Here are some known results :

- $p_\xi(n) \leq n$ for some n if and only if ξ is ultimately periodic, and in this case the complexity is bounded [13];

Research supported by NSFC. 11171123 & 11222111.

- A sequence ξ of complexity $p_\xi(n) = n + 1$ is called Sturmian. There are many equivalent characterizations and interesting properties of Sturmian sequences (see, e.g. [9, 18, 22]);
- Rote [17] constructed a class of sequences with complexity $2n$ by using graphs;
- Mossé [14] studied the case of q -automata (which correspond to substitutions of constant length). A method to compute $p(n)$ with linear recurrence formula was given under some technical conditions;
- Over a ternary alphabet, a class of Tribonacci type substitutions with complexity $2n + 1$ was introduced by Arnoux and Rauzy [3]. An example of substitution (Triplex Substitution) with complexity $3n$ is presented by the authors [21].
- For a fixed point of some substitution, the complexity can only be of the following five different asymptotic forms: $\Theta(1)$, $\Theta(n)$, $\Theta(n \log \log n)$, $\Theta(n \log n)$ or $\Theta(n^2)$, where $\Theta(g(n))$ means a function $f(n)$ satisfying $0 < \liminf \frac{f(n)}{g(n)} \leq \limsup \frac{f(n)}{g(n)} < \infty$ [15].
- For a survey and more general computation of factor complexity of word (on a alphabet of cardinality more than 2), we suggest to see [6, 8].

In this paper, we consider general substitutions σ over a binary alphabet. Using Mossé's theory of identifiability ([14]) and by studying various types of special words ([5, 6]), we show that the complexity $p(n)$ can be completely formulated knowing some initial values, and a recurrence formula is given.

2. NOTATIONS AND PRELIMINARY

We fix the binary alphabet $\mathbb{A} = \{a, b\}$ consisting of two letters a and b . Let \mathbb{A}^* be the free monoid generated by \mathbb{A} (with the empty word ε as the neutral element), and $\mathbb{A}^{\mathbb{N}}$ be the set of all infinite sequences (also called infinite words) over \mathbb{A} .

If $w \in \mathbb{A}^*$, we denote by $|w|$ its length and by $|w|_a$ (resp. $|w|_b$) the number of occurrences of the letter a (resp. b) in w . The abelian Parikh vector of w is then defined to be the column vector $L(w) = (|w|_a, |w|_b)^t \in \mathbb{N}^2$.

A word v is a factor of a word w (written as $v \in w$) if there exist $u, u' \in \mathbb{A}^*$, such that $w = uvu'$. It is sometimes convenient to use the notation “ \otimes ” to stand for some word which we don't care so much. Thus v is a factor of a word w if and only if $w = \otimes v \otimes$ (remark that even within a formula, \otimes 's may represent different words). We say that v is a prefix (resp. a suffix) of w if $w = v \otimes$ (resp. $w = \otimes v$), and then we write $v \triangleleft w$ (resp. $v \triangleright w$). Two words v and w are said to be comparable, written $v \bowtie w$, if either $v \triangleright w$ or $w \triangleright v$. The notions of factor and prefix extend to infinite words in a natural way.

It is also convenient to put, e.g. $\mathbb{A}^*v := \{xv; x \in \mathbb{A}^*\}$, $\mathbb{A}^*v\mathbb{A}^* := \{xvy; x, y \in \mathbb{A}^*\}$, etc. Thus $w \in \mathbb{A}^*v\mathbb{A}^* \Leftrightarrow v \in w$; $w \in v\mathbb{A}^* \Leftrightarrow v \triangleleft w$, and so on.

When $\xi = w_1 \cdots w_m \cdots \in \mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$ ($w_i \in \mathbb{A}$), we also write $\xi|_1 = w_1, \dots, \xi|_m = w_m, \dots$, and $\xi[i, j] = w_i w_{i+1} \cdots w_j$ ($i \leq j$).

As already defined, a substitution σ over \mathbb{A} is a morphism σ of \mathbb{A}^* . The matrix $M = (L(\sigma(a)), L(\sigma(b)))$ is called the incidence matrix of σ . The characteristic polynomial $\lambda^2 - \text{tr}(M)\lambda + \det(M)$ of M is also called the characteristic polynomial of σ .

If $\sigma(a)$ and $\sigma(b)$ have distinct first letters, we say that the substitution σ is marked, and if moreover $\sigma(a) = a\circledast$ and $\sigma(b) = b\circledast$, we say that σ is well-marked. It is easy to see that σ^2 is well-marked if σ is marked.

In this paper, all substitutions are assumed to be non-erasing, that is, the image of each letter is not empty. Whence, the substitution can be extended naturally to $\mathbb{A}^{\mathbb{N}}$. An infinite word $\xi = \xi_1 \xi_2 \cdots$ is a fixed point of σ if $\sigma(\xi) = \xi$.

Hereafter, we suppose that the substitution σ is primitive (i.e. its incidence matrix M is primitive: M^n possesses positive coordinates for some positive integer n). The following easy facts for a primitive substitution σ are well known:

- (1) the fixed point of σ is recurrent, that is, every factor will occur for infinitely many times; and all the fixed points of σ have the same language;
- (2) a substitution σ and its powers σ^n ($n \geq 1$) have the same fixed points, and thus have the same language;
- (3) if one substitution is a composition of an inner automorphism (of the free group) with another substitution, then the two substitutions have the same language.

We suppose also that the fixed point ξ of σ is not (ultimately) periodic; the periodic case are characterized completely by Séébold [19]. In particular, whence $\{\sigma(a), \sigma(b)\}$ is a code, and thus σ is marked up to an inner automorphism (see [9]). For the sake of calculation of the complexity of a non-periodic primitive substitution, we may further suppose, without loss of generality, that the substitution is well-marked.

The notion of “special words” is a powerful tool for calculating the complexity. See [5, 6] and [4, 9, 10] for more information.

Let W be a factor of ξ . If $\delta \in \mathbb{A}$ such that $W\delta$ is a factor of ξ , then we say that $W\delta$ is a right extension of W . A word is called a right special word (special word for short) of ξ if it has more than one extensions, that is, $Wa \in \xi$ and $Wb \in \xi$. Similarly we define “left extension” and “left special word”. It is easy to see that a suffix (resp. prefix) of a special (resp. left special) word is also special (resp. left special).

Let \mathcal{S}_n (resp. \mathcal{LS}_n) be the set of special words (resp. left special words) of length n of ξ . Put $\mathcal{S} = \cup_{n \geq 0} \mathcal{S}_n$ (resp. $\mathcal{LS} = \cup \mathcal{LS}_n$). It is easy to see that

$$s(n) := \#\mathcal{S}_n = \#\mathcal{LS}_n = \Delta p(n+1) (= p(n+1) - p(n)).$$

Hence the study of $p(n)$ is almost equivalent to the study of $s(n)$.

2.1. The word W_0 and the letters δ_a, δ_b .

Write $A = \sigma(a), B = \sigma(b)$, and denote $\{A, B\}^*$ the set of words obtained by a finite concatenation of the words A and B . Put, as before, e.g. $\{A, B\}^*A := \{VA; V \in \{A, B\}^*\}$. Remark that since σ is non-periodic, $\{A, B\}$ is a code and $\{A, B\}^*$ is a disjoint union of $\{A, B\}^*A$ and $\{A, B\}^*B$.

Since σ is non-periodic, the left-infinite words $A^\infty (= \cdots AA \cdots A)$ and B^∞ are different. Let W_0 be the longest common suffix of A^∞ and B^∞ (see also [20]). Remark that W_0 is possibly empty.

The following lemma is a direct consequence of Fine-Wilf theorem [16].

Lemma 2.1. $|W_0| \leq |A| + |B| - 2$.

By the definition of W_0 , for some $\delta_a, \delta_b \in \{a, b\}$ with $\{\delta_a, \delta_b\} = \{a, b\}$,

$$(2.1) \quad A^\infty = \otimes \delta_a W_0 \quad \text{and} \quad B^\infty = \otimes \delta_b W_0.$$

Formula (2.1) shows that there exist $m \geq 0$ and $A' \triangleright A$ ($|A'| < |A|$) such that

$$(2.2) \quad W_0 = A' A^m, \quad \text{and} \quad \delta_a A' \triangleright A,$$

and similarly

$$W_0 = B' B^k, \quad \text{and} \quad \delta_b B' \triangleright B.$$

The following lemma is essentially due to [20].

Lemma 2.2. (1) For $W \in \{A, B\}^*$, we have $W_0 \bowtie W$. Furthermore,

(2) If $W \in \{A, B\}^*A$ (resp. $\{A, B\}^*B$) and $|W| > |W_0|$, then $\delta_a W_0 \triangleright W$ (resp. $\delta_b W_0 \triangleright W$), where δ_a and δ_b are defined in (2.1).

(3) Let $W \in \{A, B\}^*$. If $\delta_a W_0 \triangleright W$ (resp. $\delta_b W_0 \triangleright W$), then $W \in \{A, B\}^*A$ (resp. $\{A, B\}^*B$).

In brief, any word in $\{A, B\}^*$ is comparable with W_0 . Amongst them, the word in $\{A, B\}^*A$ is comparable with $\delta_a W_0$ and $\{A, B\}^*B$ is comparable with $\delta_b W_0$.

Proof. If $W = A$ or $W = B$, the lemma is obvious. Suppose $W \in \{A, B\}^*$ such that $W_0 \bowtie W$, we claim that $\delta_a W_0 \bowtie WA$ and $\delta_b W_0 \bowtie WB$. The two statements can be proven in the same way, and we only show the first one by considering the following two cases:

Case 1: $W_0 \triangleright W$. Then $W_0 A \triangleright WA$, and on the other hand, $\delta_a W_0 \triangleright W_0 A$ because both of them are suffixes of A^∞ . Hence $\delta_a W_0 \triangleright WA$.

Case 2: $W \triangleright W_0$. Then $WA \triangleright W_0 A$, while $W_0 A$ is a suffix of A^∞ , and thus WA is a suffix of A^∞ . This yields that $WA \bowtie \delta_a W_0$ because both of them are suffixes of A^∞ . \square

Corollary 2.1. Let $W \in \{A, B\}^*$. Then $W_0 \triangleright W_0 W$, $\delta_a W_0 \triangleright W_0 WA$, $\delta_b W_0 \triangleright W_0 WB$. In particular, $\delta_a W_0 \triangleright W_0 A$, $\delta_b W_0 \triangleright W_0 B$.

2.2. Natural decomposition and identifiability.

Let ξ be a fixed sequence of σ . Write $\xi = \xi_1 \xi_2 \cdots$. Since $\sigma(\xi) = \xi$, we have the following so called “natural decomposition” of ξ

$$(2.3) \quad \xi = [\xi_1 \xi_2 \cdots \xi_{n_2-1}] [\xi_{n_2} \cdots \xi_{n_3-1}] \cdots [\xi_{n_k} \cdots \xi_{n_{k+1}-1}] [\xi_{n_{k+1}} \cdots ,$$

where $\xi_k \in \mathbb{A} = \{a, b\}$, $\sigma(\xi_k) = \xi_{n_k} \cdots \xi_{n_{k+1}-1} \in \{A, B\}$ ($k \geq 1$), and $n_1(=1), \cdots, n_k(=|\sigma(\xi[1, k-1])|+1), \cdots$ are called the “cutting positions” of ξ . We denote

$$(2.4) \quad E_1 = \{n_k; k \geq 1\}.$$

Now consider the factors of ξ . Let $W = \xi_i \xi_{i+1} \cdots \xi_j \in \xi$, then (comparing to (2.3)) for some integers k, l ($n_{k-1} < i \leq n_k \leq n_l \leq n_{l+1} - 1 \leq j < n_{l+2}$), we have

$$W = \xi_i \cdots \xi_{n_k-1} [\xi_{n_k} \cdots \xi_{n_{k+1}-1}] \cdots [\xi_{n_l} \cdots \xi_{n_{l+1}-1}] [\xi_{n_{l+1}} \cdots \xi_j,$$

that is, observing the cutting positions of W in ξ we can write out the following natural decomposition of W

$$(2.5) \quad W = U\sigma(\xi_k) \cdots \sigma(\xi_l)V = U\sigma(W')V,$$

where

$$\begin{aligned} U &= \xi_i \cdots \xi_{n_k-1} \triangleright \sigma(\xi_{k-1}), \quad |U| < |\sigma(\xi_{k-1})|, \\ \sigma(\xi_k) &= \xi_{n_k} \cdots \xi_{n_{k+1}-1}, \\ &\vdots \\ \sigma(\xi_l) &= \xi_{n_l} \cdots \xi_{n_{l+1}-1}, \\ V &= \xi_{n_{l+1}} \cdots \xi_j \triangleleft \sigma(\xi_{l+1}), \quad |V| < |\sigma(\xi_{l+1})|, \\ W' &= \xi_k \cdots \xi_l \in \xi. \end{aligned}$$

We say that W' (resp. ξ_m , $k \leq m \leq l$) is the ancestor of $\sigma(W')$ (resp. $\sigma(\xi_m)$). Sometimes, we also call $\xi_{k-1}\xi_k \cdots \xi_l\xi_{l+1}$ the ancestor of W .

We extend a little more the significance of “natural decomposition”: if $W = U\sigma(W_1W''W_2)V$ as in (2.5), we shall also say that $W = U'\sigma(W'')V'$ is a “natural decomposition” (where $U' = U\sigma(W_1)$, $V' = \sigma(W_2)V$), and we write $W = U'[\sigma(W'')]V'$. Equivalently, the notation $U'[\sigma(W'')]V'$ means that there exist $U'', V'' \in \mathbb{A}^*$ such that

$$(2.6) \quad U''W''V'' \in \xi, \quad U' \triangleright \sigma(U''), \quad \text{and} \quad V' \triangleleft \sigma(V'').$$

Intuitively, $U'[\sigma(W'')]V'$ appears in ξ with “[” and “] ” showing the interested natural cutting positions.

We call the decomposition as in (2.5) a strict natural decomposition of W . Remark that any natural decomposition can be extended to a strict one, and, in general, the natural decompositions of a factor are not unique; and that the fact $U\sigma(W)V \in \xi$ does not always mean $U[\sigma(W)]V$!

From the theory of identifiability we have (recall that $\xi[i, j] = \xi_i \cdots \xi_j$):

Lemma 2.3. [14] *There exists an integer C (depending on σ) such that, if $W \in \xi$ can be written as $W = \xi[i - C, i + C] = \xi[j - C, j + C]$ with $i \in E_1$, then we have $j \in E_1$.*

We shall say that $\xi[i - C, i + C]$ and $\xi[j - C, j + C]$ have a relative common cutting position (at the positions i and j respectively). As a consequence, if W is long enough, say $|W| \geq L$ with

$$(2.7) \quad L = \max\{2C + \max\{|A|, |B|\}, |A| + |B| - 1\} (> |W_0|)$$

and it appears at different positions in ξ : $W = \xi[i_1, i_2] = \xi[j_1, j_2]$, then roughly speaking, at the middle position of $\xi[i_1, i_2]$ and $\xi[j_1, j_2]$, they have a relative common cutting position: for some integer $N \in (|W|/2 - \max\{|A|, |B|\}, |W|/2 + \max\{|A|, |B|\})$, $i_1 + N \in E_1$ and $j_1 + N \in E_1$.

3. THE OPERATOR T AND STRUCTURE OF \mathcal{LS}

Define $T : \mathbb{A}^* \rightarrow \mathbb{A}^*$:

$$T(W) = W_0 \sigma(W).$$

Notice that T is not a morphism on \mathbb{A}^* . It is readily checked that T is injective and

$$(3.8) \quad T^n(W) = W_0 \sigma(W_0) \cdots \sigma^{n-1}(W_0) \sigma^n(W).$$

Lemma 3.1. *If $W \in \xi$, then $T(W) \in \xi$. Moreover, $T(W) = W_0[\sigma(W)]$.*

Proof. Due to the primitivity of σ , the fixed sequence ξ is recurrent. Thus for any $n \in \mathbb{N}$, $UW \in \xi$ for some $U \in \mathbb{A}^*$ with $|U| = n$. Now by the σ -invariance of ξ , we have that $\sigma(U)\sigma(W) \in \xi$. When the length n of U is large, $W_0 \triangleright \sigma(U)$ by Lemma 2.2, therefore $T(W) = W_0[\sigma(W)] \in \xi$. \square

Lemma 3.2. *Let $W_1, W_2 \in \mathbb{A}^*$. Then $T(W_1) = T(W_2)$ if and only if $W_1 = W_2$; $T(W_1) \triangleleft T(W_2)$ if and only if $W_1 \triangleleft W_2$; $T(W_1) \triangleright T(W_2)$ if and only if $W_1 \triangleright W_2$.*

Proof. The first two easy statements hold since σ is well marked, and the last one follows from Corollary 2.1. \square

The following lemma tells us that if a factor W appears at two positions with different natural decompositions, then, up to a prefix $W'_0 \triangleright W_0$, they have the same relative cutting positions.

Lemma 3.3. *Suppose that $W \in \xi$, $|W| \geq L$ with L defined in (2.7), and that W appears at two different positions in ξ , with $W = P_1[\sigma(U_1)]Q_1$ and $W = P_2[\sigma(U_2)]Q_2$ the corresponding strict natural decompositions. Then, denoting by U the longest common suffix of U_1 and U_2 and thus writing $U_1 = U'_1 U$, $U_2 = U'_2 U$ (where U'_1 or U'_2 is possibly empty), we have that U is nonempty and*

$$(3.9) \quad P_1 \sigma(U_1) Q_1 = W'_0 [\sigma(U)] Q = P_2 \sigma(U_2) Q_2,$$

where $Q = Q_1 = Q_2$, $W'_0 = P_1\sigma(U'_1) = P_2\sigma(U'_2) \bowtie W_0$. More precisely, either $W'_0 \triangleright W_0$, or $U'_1 = U'_2 = \epsilon$ and $W'_0 \triangleright \sigma(\delta)$ for some $\delta \in \mathbb{A}$.

Proof. By Lemma 2.3, the two strict natural decompositions share a relative cutting position, and thus all the cutting positions after this one. This implies that U_1 and U_2 have nonempty common suffix, i.e., U is not empty. Also this implies that $Q_1 = Q_2$, and consequently that $P_1\sigma(U'_1) = P_2\sigma(U'_2) \bowtie W_0$, where the last formula is due to Lemma 2.2. \square

Lemma 3.4. (1) If $W \in \mathcal{LS}$ with $|W| \geq L$. Then there exist unique $U \in \mathbb{A}^*$, $\delta \in \mathbb{A}$ and $Q \triangleleft \sigma(\delta)$ with $U\delta \in \xi$ and $|Q| < |\sigma(\delta)|$, such that

$$aW = aW_0[\sigma(U)]Q \quad \text{and} \quad bW = bW_0[\sigma(U)]Q.$$

(2) If $W \in \mathcal{S}$ with $|W| \geq L$. Then there exist $U \in \mathbb{A}^*$, $W'_0 \in \mathbb{A}^*$ with either $W'_0 \triangleright W_0$, or $W'_0 \triangleright \sigma(\delta)$ and $|W'_0| < |\sigma(\delta)|$ for some $\delta \in \mathbb{A}$, such that

$$Wa = W'_0[\sigma(U)]a \quad \text{and} \quad Wb = W'_0[\sigma(U)]b.$$

(3) If $W \in \mathcal{LS} \cap \mathcal{S}$ with $|W| \geq L$. Then there exists a unique $U \in \mathbb{A}^*$ such that $W = T(U)$.

Remark: The word w in $\mathcal{LS} \cap \mathcal{S}$ is called a bispecial word, which is developed in [5], see also [4].

Proof. (1) Consider the strict natural decompositions of aW and bW :

$$aW = aP_a[\sigma(U_a)]Q_a \quad \text{and} \quad bW = bP_b[\sigma(U_b)]Q_b,$$

with U the longest common suffix of U_a and U_b , $U_a = U'_a U$, $U_b = U'_b U$. Then, as in the previous proof, U is nonempty, $Q_a = Q_b$, $P_a\sigma(U'_a) = P_b\sigma(U'_b)$. Moreover, putting $W'_0 = P_a\sigma(U'_a)$, we have that $aW'_0 \triangleright \sigma(W_a)$ and $bW'_0 \triangleright \sigma(W_b)$ with $W_a, W_b \in \mathbb{A}^*$ and the last letters of W_a and W_b are distinct. Together with Lemma 2.2, these facts imply that $W'_0 = W_0$.

(2) The proof for this part is similar to the first part.

(3) This is a corollary of the first two parts. \square

Lemma 3.5. (1) $W_0 \in \mathcal{LS}$;

(2) Any prefix of a left special word is left special;

(3) If $W \in \mathcal{LS}$, then $T(W) \in \mathcal{LS}$.

(4) Let $W \in \mathcal{LS}$ with $|W| \geq L$, then there exist unique $U \in \xi$, $\delta \in \{a, b\}$ such that $W = W_0[\sigma(U)]Q = T(U)Q \triangleleft T(W')$ (see Lemma 3.4), where $W' = U\delta$. Further more, $U, W' \in \mathcal{LS}$.

Proof. (1) and (2) are obvious.

(3). If $aW \in \xi$, then $T(aW) \in \xi$ by Lemma 3.1. By Lemma 2.2, $\delta_a T(W) = \delta_a W_0 \sigma(W)$ is a suffix of $T(aW) = W_0 A \sigma(W)$, and thus $\delta_a T(W) \in \xi$. From this, we see that $W \in \mathcal{LS}$ implies $T(W) \in \mathcal{LS}$.

(4). It follows from the proof of the preceding lemma. \square

Now let

$$\overline{\mathcal{LS}} = \bigcup_{i=1}^L \mathcal{LS}_i, \quad \overline{\mathcal{LS}}_n = \{W; W \triangleleft T^n(W'), W' \in \overline{\mathcal{LS}}\}.$$

Remark that $\overline{\mathcal{LS}}_n$ is monotone with respect to n . The following theorem follows directly from the above lemma:

Theorem 3.1. $\mathcal{LS} = \bigcup \overline{\mathcal{LS}}_n = \lim_{n \rightarrow \infty} \overline{\mathcal{LS}}_n$.

Remark: The above theorem tells us that all left special words (which determine the complexity) can be obtained from a finite set $\overline{\mathcal{LS}}$ of left special words and by the operation T .

4. STRUCTURE OF \mathcal{S} AND CALCULATION OF $\Delta^2 p(n)$

Knowing the initial values, calculating $p(n)$ boils down into calculating $\Delta s(n+1) = \#\mathcal{S}_{n+1} - \#\mathcal{S}_n$. Notice that any suffix of a special word is also special, hence if $W \in \mathcal{S}_{n+1}$ then $W = \delta W'$ for some $W' \in \mathcal{S}_n$ and $\delta \in \{a, b\}$. Thus the set of special words can be visualized as a tree showing clearly how \mathcal{S}_{n+1} derives from \mathcal{S}_n (see the example and the figure therein in the last section).

As usual, for studying the special words' tree, we shall use the following notations for special words, see also [6]:

Definition 4.1. Let $W \in \mathcal{S}$. If neither aW nor bW is in \mathcal{S} , we say that W is a weak special word; If both aW and bW are in \mathcal{S} , we say that W is a strong special word. We denote by \mathcal{S}^0 and \mathcal{S}^2 the set of weak special words and the strong weak special words respectively. The collection of other special words is denoted by \mathcal{S}^1 .

For $i \in \{0, 1, 2\}$, we write $\mathcal{S}_n^i = \mathcal{S}^i \cap \mathcal{L}_n$. It is clear that

$$\mathcal{S}_n = \mathcal{S}_n^0 \cup \mathcal{S}_n^1 \cup \mathcal{S}_n^2 \text{ and } \mathcal{S} = \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2.$$

Lemma 4.1. (1) $\Delta s(n+1) = s(n+1) - s(n) = \#\mathcal{S}_n^2 - \#\mathcal{S}_n^0$.

(2) $\mathcal{S}_n^0 \cup \mathcal{S}_n^2 \subset \mathcal{S}_n \cap \overline{\mathcal{LS}}_n$.

Proof. (see Theorem 4.5.4 [6]) (1) and the fact that $\mathcal{S}_n^2 \subset \overline{\mathcal{LS}}_n$ are obvious. If a special word has only one left extension, then this left extension is also special. \square

Lemma 4.2. Let $c, d \in \mathbb{A}, W \in \xi$. If $cWd \in \xi$, then $\delta_c T(W)d \in \xi$. Conversely, if $\delta_c T(W)d \in \xi$ and $|T(W)| \geq L$, then $cWd \in \xi$.

Proof. If $cWd \in \xi$, then by Lemma 3.1, $T(cWd) \in \xi$, i.e., $W_0 \sigma(c) \sigma(W) \sigma(d) \in \xi$. This together with Corollary 2.1 and the fact that σ is well marked implies that $\delta_c W_0 \sigma(W)d = \delta_c T(W)d \in \xi$.

Conversely, if $\delta_c T(W)d \in \xi$ and $|T(W)| \geq L$, then by Lemma 3.3, we know that $\delta_c T(W)d = \delta_c W_0 [\sigma(W)]d$ is a natural decomposition. Considering the ancestor of $\delta_c T(W)d$, we know, again by Corollary 2.1 and the fact that σ is well marked, that $cWd \in \xi$. \square

Lemma 4.3. *If $W \in \mathcal{S}$, then $T(W) \in \mathcal{S}$ (thus $\sigma(W) \in \mathcal{S}$); furthermore $T(W)a = W_0[\sigma(W)]a$, and $T(W)b = W_0[\sigma(W)]b$.*

Conversely if $W \in \mathcal{S}$ and $|W| \geq L$, then there exists $U \in \mathcal{S}$ such that $W \triangleright T(U)$.

Proof. Let $W \in \mathcal{S}$, then $Wa, Wb \in \xi$, and by Lemma 3.1,

$$W_0[\sigma(W)]A, W_0[\sigma(W)]B \in \xi.$$

Recalling $A = a \otimes$ and $B = b \otimes$, The first part of our lemma is thus proved.

The rest part is a restatement of Lemma 3.4(2). \square

We can say more on the structure of \mathcal{S}^2 and \mathcal{S}^0 .

Lemma 4.4. *If $W \in \mathcal{S}^2$ then $T(W) \in \mathcal{S}^2$. Conversely if $W \in \mathcal{S}^2$ and $|W| \geq L$, then there exists a unique $U \in \mathcal{S}^2$ such that $W = T(U)$.*

Proof. Let $W \in \mathcal{S}^2$. Then we have, by definition, that

$$(4.10) \quad aWa, aWb, bWa, bWb \in \xi,$$

and, by Lemma 4.2, that

$$\delta_a T(W)a, \delta_a T(W)b, \delta_b T(W)a, \delta_b T(W)b \in \xi,$$

i.e., $T(W) \in \mathcal{S}^2$. The first part of the lemma is proved.

Now suppose $W \in \mathcal{S}^2$ and $|W| \geq L$. Then by Lemmas 4.1(2) and 3.4(3), $W = T(U)$. By Lemma 4.2, $U \in \mathcal{S}^2$. \square

Lemma 4.5. *If $W \in \mathcal{S}^0$ and $|T(W)| \geq L$, then $T(W) \in \mathcal{S}^0$. Conversely if $W \in \mathcal{S}^0$ and $|W| \geq L$, then there exists a unique $U \in \mathcal{S}^0$ such that $W = T(U)$.*

Proof. By Lemma 4.2, when $|T(W)| \geq L$ we know that $cWd \in \xi$ if and only if $\delta_c T(W)d \in \xi$. Whence $W \in \mathcal{S}^0$ if and only if $T(W) \in \mathcal{S}^0$. The remaining proof is almost same with the corresponding part for the preceding Lemma. \square

Now denote $\overline{\mathcal{S}^2} = \bigcup_{i=1}^L \mathcal{S}_i^2$ the set of strong special words of length less than L ; $\widetilde{\mathcal{S}^2}$ the set of the words $W \in \overline{\mathcal{S}^2}$ such that $|T(W)| > L$. The sets $\overline{\mathcal{S}^0}$ and $\widetilde{\mathcal{S}^0}$ are defined in a similar way. Let

$$(4.11) \quad \widetilde{\mathcal{S}} = \widetilde{\mathcal{S}^0} \cup \widetilde{\mathcal{S}^2}$$

which will be considered as “initial special words”.

Lemma 4.6. *For any $n > L$, we have*

$$\#\mathcal{S}_n^2 = \sum_{W \in \widetilde{\mathcal{S}^2}} \sum_{k \geq 1} \delta(|T^k(W)|, n), \quad \text{and} \quad \#\mathcal{S}_n^0 = \sum_{W \in \widetilde{\mathcal{S}^0}} \sum_{k \geq 1} \delta(|T^k(W)|, n),$$

where $\delta(i, j)$ is the Kronecker symbol: $\delta(i, j) = 1$ if $i = j$ and $= 0$ otherwise.

Proof. Let $U \in \mathcal{S}_n^2$. By Lemma 4.4, there exist $k \geq 1$ and $W \in \widetilde{\mathcal{S}}^2$, which are unique, such that $U = T^k(W)$. Conversely if $|T^k(W)| = n$ for some $k \geq 1, W \in \widetilde{\mathcal{S}}^2$, then $T^k(W) \in \mathcal{S}_n^2$. Thus we have

$$\mathcal{S}_n^2 = \{U; U = T^k(W), |T^k(W)| = n, k \geq 1, W \in \widetilde{\mathcal{S}}^2\}$$

where k and W in the representation $U = T^k(W)$ are uniquely determined by U . The first equality is thus proved. The second is proved similarly. \square

The following formula then follows from the above lemma and Lemma 4.1:

Lemma 4.7. *For any $n > L$, we have*

$$\begin{aligned} \Delta s(n+1) &= s(n+1) - s(n) \\ &= \sum_{W \in \widetilde{\mathcal{S}}^2} \sum_{k \geq 1} \delta(|T^k(W)|, n) - \sum_{W \in \widetilde{\mathcal{S}}^0} \sum_{k \geq 1} \delta(|T^k(W)|, n). \end{aligned}$$

It can be written as

$$\Delta s(n+1) = \sum_{W \in \overline{\mathcal{S}}} \sum_{k \geq 1} \text{sgn}(W) \delta(|T^k(W)|, n),$$

where $\overline{\mathcal{S}} = \bigcup_{i=1}^L \mathcal{S}_i$ (the special words of length less than L), and

$$(4.12) \quad \text{sgn}(W) = \begin{cases} -1 & \text{if } W \in \widetilde{\mathcal{S}}^0 \\ 1 & \text{if } W \in \widetilde{\mathcal{S}}^2 \\ 0 & \text{otherwise.} \end{cases}$$

Remark: 1. The function $\text{sgn}(\cdot)$ is equal to the bilateral multiplicity of a factor ([6]). See Theorem 4.5.4 [6] for more general cases.

2. The above lemma tells us that the complexity $p(n)$ can be computed knowing a finite set $\overline{\mathcal{S}}$ of special words. In the next section, we will find out a (non-linear) recurrence formula for the computation.

5. RECURRENCE FORMULA FOR THE COMPLEXITY

Recall that M denotes the incidence matrix of σ . Then M^2 is the incidence matrix of σ^2 which possess non-negative eigenvalues. Since σ and σ^2 share the fixed sequence ξ , we may suppose without loss of generality that the eigenvalues of M is non-negative.

Let $\lambda_1 \geq \lambda_2 \geq 0$ be the two eigenvalues, V_1, V_2 be the corresponding eigenvectors. Since M is primitive, $\lambda_1 > \lambda_2$ and V_1 is positive.

Recall that: for $W \in \{a, b\}^*$, $L(W) = (|W|_a, |W|_b)^t$,

$$(5.13) \quad |\sigma^n(W)| = (1, 1)M^n L(W).$$

Lemma 5.1. *Let $X, Y \in \mathbb{R}^2$. Then there exists $N = N(X, Y) \geq 1$ such that $(1, 1)M^{N+n}(X - Y)$ ($n \in \mathbb{N}$) is of constant sign. That is,*

$$(1, 1)M^{N+n}X > (\text{resp. } =, <) (1, 1)M^{N+n}Y \text{ for all } n \in \mathbb{N}.$$

Proof. Let $X - Y = \mu_1 V_1 + \mu_2 V_2$ where $\mu_1, \mu_2 \in \mathbb{R}$, then for $k \geq 1$,

$$(1, 1)M^k(X - Y) = \lambda_1^k \mu_1 (1, 1)V_1 + \lambda_2^k \mu_2 (1, 1)V_2.$$

Case 1. $\mu_1 = 0$. Then $(1, 1)M^k(X - Y) = \lambda_2^k \mu_2 (1, 1)V_2$, which is obviously of the sign of $\lambda_2 \mu_2 (1, 1)V_2$ independent of $k \geq 1$.

Case 2. $\mu_1 > 0$. Since $\lambda_1 > 0$, $(1, 1)V_1 > 0$ and $\lambda_1 > \lambda_2 \geq 0$, there exists $N \geq 1$ such that for $k \geq N$ we have $\lambda_1^k \mu_1 (1, 1)V_1 + \lambda_2^k \mu_2 (1, 1)V_2 > 0$.

Case 3. $\mu_1 < 0$. The similar proof as Case 2. \square

Corollary 5.1. *Let $W_1, W_2 \in \mathbb{A}^*$. There exists $N = N(W_1, W_2)$ such that $|T^{N+n}(W_1)| - |T^{N+n}(W_2)|$ ($n \in \mathbb{N}$) is of constant sign. This sign (called the final sign) will be denoted by $\text{SGN}\{W_1, W_2\}$.*

Proof. The lemma follows directly from the above lemma and (5.13). \square

In fact, we can say more:

Corollary 5.2. *Let $W_1, W_2 \in \mathbb{A}^*$. Then there exist $m_1 = m_1(W_1, W_2), m_2 = m_2(W_1, W_2) \in \mathbb{N}$ such that one of the following alternatives holds:*

- (1). $|T^{m_1}(W_1)| = |T^{m_2}(W_2)| < |T^{m_1+1}(W_1)| = |T^{m_2+1}(W_2)| < |T^{m_1+2}(W_1)| = |T^{m_2+2}(W_2)| < \dots$
- (2). $|T^{m_1}(W_1)| < |T^{m_2}(W_2)| < |T^{m_1+1}(W_1)| < |T^{m_2+1}(W_2)| < |T^{m_1+2}(W_1)| < |T^{m_2+2}(W_2)| < \dots$

Proof. If $\text{SGN}(T^m(W_1), T^n(W_2)) = 0$ for some $m, n \in \mathbb{N}$, the alternative (1) holds.

Otherwise, $\text{SGN}(T^m(W_1), T^n(W_2)) \neq 0$ for any $m, n \in \mathbb{N}$. We assume, without loss of generality, that $\text{SGN}(W_1, W_2) = -1$. Due to the primitivity, W_2 is a factor of $T^l(W_1)$ for l large enough, and it turns out that $\text{SGN}(T^l(W_1), W_2) = 1$. Now clearly $m \mapsto \text{SGN}(T^m(W_1), W_2)$ is an increasing mapping from \mathbb{N} onto $\{-1, 1\}$, therefore there exists $m \in \mathbb{N}$ such that $\text{SGN}(T^m(W_1), W_2) = -1$, while $\text{SGN}(T^{m+1}(W_1), W_2) = 1$. Whence the alternative (2) holds for

$$m_2 = \max\{N(T^m(W_1), W_2), N(T^{m+1}(W_1), W_2)\}, \text{ and } m_1 = m + m_2. \quad \square$$

Now we can deduce from the above lemma the recurrence properties of the complexity. First let $\tilde{\mathcal{S}} = \{S_1, S_2, \dots, S_K\}$ and denote

$$n_1 - 1 = \max \left\{ \max\{m_1(W_1, W_2), m_2(W_1, W_2)\}; \quad W_1, W_2 \in \tilde{\mathcal{S}} \right\},$$

where $m_1(W_1, W_2), m_2(W_1, W_2)$ are defined in Lemma 5.2.

We start from $T^{n_1}(S_1)$. By Lemma 5.2, for each $j = 2, 3, \dots, K$, there exists unique $n_j \in \mathbb{N}$ such that $|T^{n_1}(S_1)| \leq |T^{n_j}(S_j)| < |T^{n_1+1}(S_1)|$. Without loss of generality we may suppose that

$$|T^{n_1}(S_1)| \leq |T^{n_2}(S_2)| \leq \dots \leq |T^{n_K}(S_K)| \leq |T^{n_1+1}(S_1)|.$$

Then for simplifying the notations let $N_k^j = |T^j(T^{n_k}(S_k))|$ ($1 \leq k \leq K$, $j \in \mathbb{N}$). We have by Lemma 5.2 the following unison property for the ‘‘jumps of $|T^i(W_k)|$ ’’:

$$\begin{aligned} & N_1^0 \leq N_2^0 \leq \dots \leq N_K^0 \\ & \leq N_1^1 \leq N_2^1 \leq \dots \leq N_K^1 \\ & \dots \dots \\ & \leq N_1^j \leq N_2^j \leq \dots \leq N_K^j \\ (5.14) \quad & \leq N_1^{j+1} \leq \dots \end{aligned}$$

Now we can formulate the recurrence formula of the complexity. Let $\chi_{[m,n]}$ denote the indicator function of the integers' interval $[m, n)$. Let $I^j = [N_1^j, N_1^{j+1})$, $j \in \mathbb{N}$. We see that I^j is the disjoint union of the subintervals $I_k^j = [N_k^j, N_{k+1}^j)$ ($j \in \mathbb{N}$, $k \in \{1, 2, \dots, K\}$), where $N_{K+1}^j = N_1(j+1)$. That is

$$[N_1^0, \infty) = \bigcup_{j=0}^{\infty} I^j, \quad I^j = \bigcup_{k=1}^K I_k^j.$$

5.1. Initial values of the complexity.

Finally let $c_k = \sum_{i=1}^K \text{sgn}(S_i) \delta(|T^{n_i}(S_i)|, |T^{n_k}(S_k)|)$ ($k = 1, \dots, K$), where $\text{sgn}(\cdot)$ is defined in (4.12). Then by Lemma 4.7, we have, $\Delta s(n+1) = c_k$ if $n = |T^{n_k}(S_k)|$ ($k = 1, \dots, K$) and $= 0$ otherwise. In other words, $n \mapsto s(n+1)$ ($n \in I^0$) is a step function with jumps c_k at $n = N_k(0)$ ($k \in \{1, 2, \dots, K\}$):

$$(5.15) \quad s(n+1) = s(N_1(0)) + \sum_{k=1}^K (c_1 + \dots + c_k) \chi_{I_k^0}(n) \quad (n \in I^0),$$

5.2. Recurrence formula of $s(\cdot + 1)$ on I^j .

Notice that $I^j = \bigcup_{k=1}^K I_k^j$ ($j \in \mathbb{N}$) can be calculated directly or by some easy recurrence formula as described in the following:

Proposition 5.1. *We have for any $W \in \mathbb{A}^*$, $n \in \mathbb{N}$,*

1. $|\sigma^{n+2}(W)| = \text{tr}(M) |\sigma^{n+1}(W)| - \det(M) |\sigma^n(W)|;$
 $|T^{n+2}(W)| = \text{tr}(M) |T^{n+1}(W)| - \det(M) |T^n(W)| + a,$
where $a = |\sigma(W_0)| - (\text{tr}(M) - 1)|W_0|$.
2. $|\sigma^n(W)| = \lambda_1^n \mu_1(1, 1) V_1 + \lambda_2^n \mu_2(1, 1) V_2$ if $L(W) = \mu_1 V_1 + \mu_2 V_2;$

$|T^n(W)| = \lambda_1^n \mu_1(1, 1) V_1 + \lambda_2^n \mu_2(1, 1) V_2 + b_n$,
 where $b_n (n \in \mathbb{N})$ is a fixed sequence given explicitly by $L(W_0)$ and M .

Proof. All the results can be deduced easily from (3.8), (5.13) and Cayley-Hamilton formula (with I denotes the identity matrix): $M^2 = \text{tr}(M) M - \det(M) I$. \square

We have just seen the recurrence properties of the intervals I_j ($j \in \mathbb{N}$). Still using Lemma 4.7 and the formula (5.14) and we see that what happens for $s(n+1)$ ($n \in I^j$, $j \in \mathbb{N}$) is recurrently the same as $s(n+1)$ ($n \in I^0$), i.e., similar to (5.15) we have proved the following

Theorem 5.1. *Let σ be a well marked, primitive, non-periodic substitution having non-negative eigenvalues. Then for $n \in [N_1^0, \infty) = \bigcup_{j=0}^{\infty} I^j$, the following recurrence formula holds:*

$$s(n+1) = s(N_1^j) + \sum_{k=1}^K (c_1 + \cdots + c_k) \chi_{I_k^j}(n) \quad (n \in I^j, j \geq 0).$$

Remark: 1. The conditions “primitive, well marked, non-periodic, having non-negative eigenvalues” are non-essential as have already mentioned.

2. $s(N_1^{j+1}) - s(N_1^j) \equiv c_1 + \cdots + c_K$ ($j \in \mathbb{N}$), which implies roughly $s(\lambda_1^n) \approx n(c_1 + \cdots + c_K)$ for large n .

3. Although the above mentioned N_1^0 can be more or less controlled in the proof of the theorem, but how to give efficiently this big integer N remains as an open problem.

Fially let us give briefly an example: consider the substitution $\sigma = (aab, ba)$ i.e., $a \mapsto aab, b \mapsto ba$.

For this substitution, we have $W_0 = \varepsilon$ and thus $T = \sigma$. The incidence matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and the characteristic polynomial is $\lambda^2 - 3\lambda + 1$. The fixed point reads

$$\xi = aabaabbbaaabaabbabaaabaabaabbbaaabaabbabaaabbbaaabaab \dots$$

The tree of the special words is depicted in Figure 1.

The weak and strong special words (here σ^0 is the identity map):

$$\mathcal{S}^0 = \{abaa, aabbbaaabaab, \dots\} = \{\sigma^n(abaa); n = 0, 1, 2, \dots\},$$

$$\mathcal{S}^2 = \{\varepsilon, a, aab, aabaabba, \dots\} = \{\varepsilon\} \cup \{\sigma^n(a); n = 0, 1, 2, \dots\}.$$

From the structure of special words, the numbers of special words $s(n)$ and the complexity $p(n)$ read

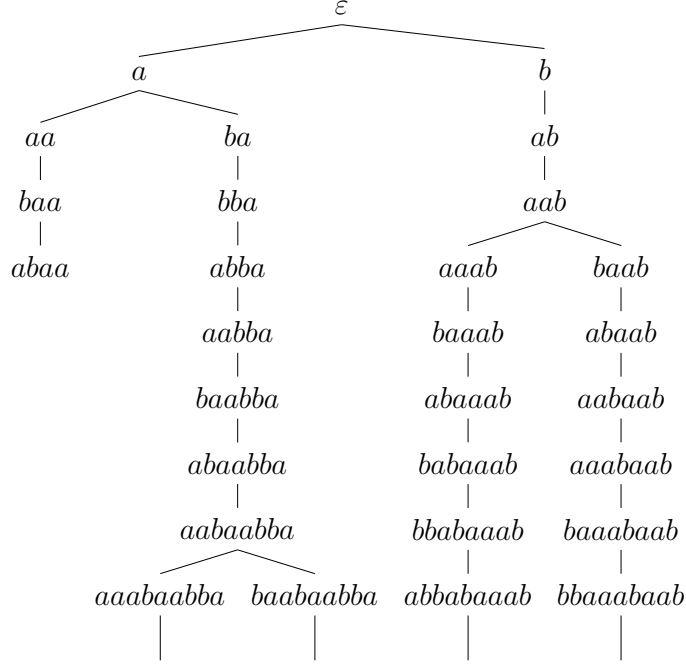


FIGURE 1. Tree of Special Words

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$s(n)$	1	2	3	3	4	3	3	3	3	4	4	4	3	3	3	3	...
$p(n)$	1	2	4	7	10	14	17	20	23	26	30	34	38	41	44	47	...

We can formulate $s(n)$ as

$$s(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ 3 & \text{if } n \in \{2, 3\} \cup \bigcup_{k \geq 0} [d(k) + 1, g(k + 1)], \\ 4 & \text{if } n \in \bigcup_{k \geq 0} [g(k) + 1, d(k)], \end{cases}$$

where the number sequences $g(k)$ and $d(k)$ are defined as

$$g(k) = (1, 1)M^k(2, 1)^t, \quad d(k) = (1, 1)M^k(3, 1)^t,$$

satisfying both the same recurrence:

$$\begin{cases} g(k + 2) = 3g(k + 1) - g(k), \\ d(k + 2) = 3d(k + 1) - d(k), \end{cases}$$

with $g(0) = 3, g(1) = 8$ and $d(0) = 4, d(1) = 11$.

ACKNOWLEDGEMENT The authors would like to thank Prof. Z.Y. Wen (Tsinghua), J.P. Allouche (Jussieu) and others for helpful discussions, references and corrections.

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SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, 430074 WUHAN, P. R. CHINA

E-mail address: `tanbo@hust.edu.cn`

SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, 430074 WUHAN, P. R. CHINA

E-mail address: `zhi-xiong.wen@hust.edu.cn`

SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, 430072 WUHAN, P. R. CHINA

E-mail address: `ypzhang@whu.edu.cn`